THE MIXED SCHMIDT CONJECTURE IN THE THEORY OF DIOPHANTINE APPROXIMATION.

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Abstract. Let $\mathcal{D} = (d_n)_{n=1}^{\infty}$ be a bounded sequence of integers such that $d_n \geq 2$ and i, j be non-negative numbers with $i + j = 1$. With the associated norm $|\cdot|_{\mathcal{D}}$ as defined by de Mathan and Teulié in [4] we prove that the set of $x \in \mathbb{R}$ which satisfy the inequality

$$\max\{|q|_{\mathcal{D}}^{1/i}, \|qx\|^{1/j}\} > \frac{c}{q}$$

for some constant $c = c(x) > 0$ and all $q \in \mathbb{N}$ has full Hausdorff dimension. In establishing this result we prove a p–adic variant of Schmidt's conjecture in simultaneous Diophantine approximation as a corollary.

1. Introduction

One of the major unproven conjectures in metric number theory is the Littlewood conjecture, first poised by Littlewood in the 1930s, which states that

$$\liminf_{q \to \infty} q\|qx\|\|qy\| = 0$$

for any pair (x, y) of real numbers where $\|\cdot\|$ denotes the distance to the nearest integer. Despite a concerted effort over the years to settle the problem, see [6] for examples of some recent results, the conjecture has so far resisted all attempts to prove it. Probably the most compelling evidence of the truth of the conjecture is due to Einsiedler, Katok and Lindenstrauss [6], who have shown that any exceptional set to Littlewood's conjecture must have Hausdorff dimension 0. Closely linked to the Littlewood conjecture is the following conjecture of Schmidt which states that for any $i, j, i', j' \geq 0$ with $i + j = 1 = i' + j'$

$$\mathbf{Bad}(i, j) \cap \mathbf{Bad}(i', j') \neq \emptyset$$

where

$$\mathbf{Bad}(i, j) := \{(x, y) \in \mathbb{R}^2 : \exists c(x, y) > 0 \text{ with } \max\{\|qx\|^{1/i}\|, qy\|^{1/j}\} > c(x, y)q^{-1} \ \forall\ q \in \mathbb{N}\}.$$

It is worthing noting that a counterexample to the conjecture of Schmidt would imply Littlewood's conjecture.

In [4], de Mathan and Teulié published a paper in which they discussed a number of interesting variations on classical problems in simultaneous Diophantine apporximation. Amongst these problems, they proposed a mixed Littlewood conjecture. Let $\mathcal{D}$ be a bounded sequence $(d_n)_{n=1}^{\infty}$ of integers no smaller than 2 and define the sequence $(D_n)_{n=0}^{\infty}$ as follows; let $D_0 := 1$

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and

$$D_n := \prod_{k=1}^{n} d_k$$

for all $n \geq 1$. Now set

$$\omega_{\mathcal{D}} : \mathbb{N} \to \mathbb{N} : q \mapsto \sup\{n \in \mathbb{N} : q \in D_n\mathbb{Z}\}$$

and

$$|q|_{\mathcal{D}} := D^{-1}_{\omega_{\mathcal{D}}(q)} = \inf\{1/e_n : q \in D_n\mathbb{Z}\}.$$

The de Mathan-Teulié conjecture is then

$$\liminf_{q\to\infty} q|q|_{\mathcal{D}}\|qx\| = 0.$$

Perhaps the case of most importance is when $\mathcal{D}$ is the constant sequence $\{p, p, p, \dots\}$ where p is an odd prime. In this case $|\cdot|_{\mathcal{D}}$ is the usual p-adic norm and the de Mathan-Teulié conjecture reduces to

$$\liminf_{q\to\infty} q|q|_p\|qx\| = 0.$$

As with Littlewood, the de Mathan-Teulié conjecture is still open, even in the p–adic case. However, there do exist some partial results. A recent result of Bugeaud, Haynes and Velani, [2], gives the following analogue of Gallagher's theorem

$$\liminf_{q\to\infty} q(\log q)^{k+1}|q|_{p_1}\dots|q|_{p_k}\|qx\| = 0, \tag{1}$$

for almost every real number x. Note that unlike the mixed Littlewood conjecture, this is a metric result in the sense that the result holds for almost all real numbers x. Bugeaud and Moshchevitin, [3] were able to show that for every prime p, the set of $\alpha \in \mathbb{R}$ satisfying

$$\liminf_{q\to+\infty} |q|_p\, q(\log q)^2\|qx\| > 0$$

has full Hausdorff dimension. Thus whilst the set of exceptions to Equation (1) is of Lebesgue measure zero it is of the same dimension as the entire unit interval.

For recent advances towards the "mixed" Littlewood conjecture, the reader is referred to Bugeaud, Drmota and Mathan, [1], or Einsiedler and Kleinbock, [5].

As there are such strong parallels with the classical problem of Littlewood, one would reasonably expect there to be a corresponding problems akin to the Schmidt conjecture. However, most of the work to date has focussed on the mixed Littlewood conjecture and very little has be done on establishing a "mixed" Schmidt conjecture. In this paper we consider such a problem .

Firstly, we need to introduce a little more notation. With $\mathcal{D}$ defined as above, let the set $\mathbf{Bad}_{\mathcal{D}}(i,j)$ be

$$\mathbf{Bad}_{\mathcal{D}}(i,j) := \left\{x \in \mathbb{R} \;:\; \exists c = c(x) > 0, \forall q \in \mathbb{N}, \max\{|q|_{\mathcal{D}}^{1/i}, ||qx||^{1/j}\} > \frac{c}{q}\right\} \tag{2}$$

where $i + j = 1$ and $i, j > 0$. In the case when $\mathcal{D} = (p, p, p \dots)$ where p is an odd prime we will write $\mathbf{Bad}_p(i,j)$ for $\mathbf{Bad}_{\mathcal{D}}(i,j)$. This is of course a "mixed" variant of the sets $\mathbf{Bad}(i,j)$ defined above.

The main result of this paper is then:

Theorem 1. *For each $k \in \mathbb{N}$ let $\mathcal{D}_k$ be a bounded sequence $(d_n)_{n \geqslant 1}$ of integer numbers not smaller than 2. Suppose that the pairs (i_k, j_k) of positive real numbers satisfy $i_k + j_k = 1$ for all $k \in \mathbb{N}$. Then*

$$\dim\left(\bigcap_{k=1}^{\infty} \mathbf{Bad}_{\mathcal{D}_k}(i_k, j_k)\right) = 1.$$

An immediate corollary to Theorem 1 is the p–adic Schmidt conjecture. More exactly, for any two pairs of positive numbers i, j, i', j' with $i + j = 1 = i' + j'$,

$$\dim\left(\mathbf{Bad}_p(i,j) \cap \mathbf{Bad}_p(i',j')\right) = 1.$$

2. Schmidt (α, β) games

The main technical device used in the proof of Theorem 1 is the idea of Schmidt games as introduced by Schmidt in [7]. We present a simplified form of the game suitable for our purposes here and some of the results related to games which will help us prove Theorem 1. For further details the reader is referred to the paper of Schmidt cited above.

Schmidt's game is played by two players, say A and B. Each player has an associated parameter, say α for A and β for B, where $0 < \alpha, \beta < 1$, with which to play the game. The game is played as follows. Initially the second player, B, chooses a closed interval $I_0 \subset \mathbb{R}$ to start the game. Then A chooses an interval $J_1 \subset I_0$ having length $|J_1| = \alpha |I_0|$, B continues by choosing a closed interval $I_1 \subset J_1$ having length $|I_1| = \beta |J_1|$. This initerval choosing process is repeated ad infinitum to construct a nested sequence of closed intervals

$$I_0 \supset J_1 \supset I_1 \supset J_2 \supset \dots.$$

Since all the intervals in the above sequence are closed, they intersect at one point, say γ.

Now given a set $S \subset \mathbb{R}$, we say that S is (α, β)-winning if A can play the game in such a way that

$$\gamma \in S$$

regardless of the moves made by B. The set $S \subset \mathbb{R}$ is called α–winning if it is (α, β)–winning for all real values $0 < \beta < 1$.

Sets which are α–winning are by necessity large. A fact proved by Schmidt in [7] is

Theorem S1. *Let α be a real number such that $0 < \alpha < 1$. Then for any α-winning set S,*

$$\dim S = 1,$$

where $\dim$ *is Hausdorff dimension.*

Schmidt was able to extend this result to any countable intersection of α–winning sets.

Theorem S2. *The intersection of any countable number of α-winning sets is again α-winning and thus of full Hausdorff dimension.*

It is these two results which we will appeal to in proving Theorem 1.

3. Proof of Theorem 1

Take any sequence $\mathcal{D} = (d_n)_{n\geqslant 1}$ of integers $d_n \geqslant 2$ and any pair (i,j) such that $i,j > 0$, $i+j=1$. We claim that $\mathbf{Bad}_{\mathcal{D}}(i,j)$ is α-winning for $\alpha = 1/4$. If this is indeed the case, then Theorems S1 and S2 imply that

$$\dim\Big(\bigcap_{k=1}^{\infty}\mathbf{Bad}_{\mathcal{D}_k}(i_k,j_k)\Big) = 1$$

and Theorem 1 follows immediately. Hence we only need to show that the above claim is true.

With this aim in mind, let $c>0$ be a fixed constant and define $\mathbf{Bad}_{\mathcal{D}}(c,i,j)$ to be the set of $x\in\mathbb{R}$ such that

$$\max\{|q|_{\mathcal{D}}^{1/i}, ||qx||^{1/j}\} > \frac{c}{q} \qquad \forall\, q\in\mathbb{N}. \tag{3}$$

It is easily seen that $\mathbf{Bad}_{\mathcal{D}}(c,i,j)\subset\mathbf{Bad}_{\mathcal{D}}(i,j)$.

The idea of the proof is as follows. For any (fixed) $\beta\in(0,1)$, we will show that there exists a positive constant $c=c(\beta)$, dependent only on β, such that the set $\mathbf{Bad}_{\mathcal{D}}(c,i,j)$ is (α,β)-winning. As $\mathbf{Bad}_{\mathcal{D}}(c,i,j)\subset\mathbf{Bad}_{\mathcal{D}}(i,j)$, we must have $\mathbf{Bad}_{\mathcal{D}}(i,j)$ also being (α,β)-winning and finally as β is arbitrary the desired result would follow.

The set $\mathbf{Bad}_{\mathcal{D}}(c,i,j)$ consists of exactly those real numbers that avoid the neighborhood

$$\Delta(r/q) := \left[\frac{r}{q}-\frac{c^j}{q^{1+j}}, \frac{r}{q}+\frac{c^j}{q^{1+j}}\right]$$

of any rational number r/q which satisfies $|q|_{\mathcal{D}} < c^i q^{-i}$. For convenience let's denote the set of such rational numbers by $\mathcal{C}$.

Define the value R to be

$$R := \frac{1}{\alpha\beta}.$$

Without loss of generality suppose that the second player starts with a line segment I_0 of sufficiently short length that $|I_0| := \tau\beta$ where $\tau\leqslant 1$. Is is then an easy task to compute the lengths of the intervals I_n and J_n. Namely,

$$|I_n| = \tau\beta R^{-n} \quad\text{and}\quad |J_n| = \tau R^{-n}.$$

Finally choose c such that

$$c^j < \frac{\tau\beta}{4} \quad\text{and}\quad c^i < \frac{1}{2R^{\frac{2}{j+1}}\tau\beta}. \tag{4}$$

We now describe the winning strategy for the first player. Essentially the first player should choose the interval J_n such that

$$J_n\cap\Delta(r/q) = \emptyset \quad \forall\, r/q\in\mathcal{C} \text{ with } q^{1+j} < R^n. \tag{5}$$

We will use induction to prove that the first player can always choose such an interval J_n.

It is obvious that the interval I_0 avoids all the neighbourhoods of numbers r/q with $q^{1+j} < R^0$ since there are no such numbers.

Assume now that the first player has chosen the intervals I_{n-1} where $k\leq n-1$ subject to the conditions given above. That is,

$$I_{n-1}\cap\Delta(r/q) = \emptyset \;\; \forall\, r/q\in\mathcal{C} \;\text{ with }\; q^{1+j} < R^{n-1}. \tag{6}$$

Now for each $I_{n-1} \subset J_{n-1}$, $|I_{n-1}| = \beta|J_{n-1}|$ we want to construct an interval $J_n \subset I_{n-1}$ of length $|J_n| = \alpha|I_{n-1}|$ such that the condition (5) is satisfied. By the inductive assumption while choosing J_n, we should only take care of the rationals from the collection

$$\mathcal{C}(n) := \{r/q \in \mathcal{C} \ : \ R^{n-1} \leqslant q^{1+j} < R^n\}.$$

Consider some number $r/q \in \mathcal{C}(n)$. Then

$$|\Delta(r/q)| = \frac{2c^j}{q^{1+j}} \leqslant 2c^j \cdot R^{-n+1} < \frac{|I_{n-1}|}{2},$$

since $4c^j < \tau\beta$ by the first inequality in (4). Therefore one can always choose an interval J_n of length $\frac{|I_{n-1}|}{4}$ which avoids $\Delta(r/q)$.

Now let's consider two rational numbers $r_1/q_1, r_2/q_2 \in \mathcal{C}(n)$. Write their denominators in a form

$$q_l = D_{k_l} q_l^*, \quad q_l \notin D_{k_l+1}\mathbb{Z}, \quad l = 1, 2.$$

Since $|q_l|_{\mathcal{D}} < c^i q_l^{-i}$, we have

$$D_{k_l} > q_l^i c^{-i} \geqslant R^{\frac{(n-1)i}{j+1}} c^{-i}.$$

Therefore we can find the lower bound for the greatest common divisor of q_1 and q_2:

$$(q_1, q_2) > c^{-i} R^{\frac{(n-1)i}{j+1}}. \tag{7}$$

Finally the distance between two rational points from $\mathcal{C}(n)$ is at least

$$\left|\frac{r_1}{q_1} - \frac{r_2}{q_2}\right| \geqslant \frac{(q_1, q_2)}{q_1 q_2} > \frac{c^{-i} R^{\frac{(n-1)i}{j+1}}}{R^{\frac{2n}{j+1}}} = c^{-i} R^{-\frac{2}{j+1}} R^{-n+1} > 2|I_{n-1}|,$$

since $c^{-i} R^{-\frac{2}{j+1}} > 2\tau\beta$ by the second inequality in (4).

Therefore the distance between two rational numbers from $\mathcal{C}(n)$ is large enough so there can be only one number $r/q \in \mathcal{C}(n)$ which can intersect I_{n-1}. However as we mentioned before we can take J_n which avoids $\Delta(r/q)$. So J_n satisfies (5) and this is the interval the first player should take. So the induction is finished.

The upshot is that for $\alpha = 1/4$ and any real positive β there exists a winning strategy for the first player in (α, β)-game. Therefore $\mathbf{Bad}_{\mathcal{D}}(i, j)$ is 1/4-winning.

Comments. Consider the nonnegative numbers $i_1, i_2, \ldots, i_m, j$ such that $i_1 + \ldots + i_k + j = 1$. Then for any collection $\mathcal{D}_1, \ldots, \mathcal{D}_k$ of sequences satisfying the properties given in § **??**, similar arguments to those given above can be applied to show that the set

$$\begin{aligned}\mathbf{Bad}(i_1, \ldots, i_k, j; \mathcal{D}_1, \mathcal{D}_2, \ldots \mathcal{D}_k) := \ & \{x \in \mathbb{R} \ : \ \exists c = c(x) > 0, \forall q \in \mathbb{N}, \\ & \max\{|q|_{\mathcal{D}_1}^{1/i_1}, |q|_{\mathcal{D}_2}^{1/i_2}, \ldots, |q|_{\mathcal{D}_k}^{1/i_k}, ||qx||^{1/j}\} > \tfrac{c}{q}\}.\end{aligned}$$

is 1/4-winning.

References

[1] Yann Bugeaud, Michael Drmota, and Bernard de Mathan, *On a mixed Littlewood conjecture in Diophantine approximation*, Acta Arith. **128** (2007), no. 2, 107–124, DOI 10.4064/aa128-2-2. MR **2313997 (2008d:**11067)

[2] Yann Bugeaud, Alan Haynes, and Sanju Velani, *Metric considerations concerning the mixed Littlewood Conjecture*, arXiv:0909.3923v1 (2009).

[3] Yann Bugeaud and Nikolay Moshchevitin, *Badly approximable numbers and Littlewood-type problems*, arXiv:0905.0830v1 (2009).

[4] Bernard de Mathan and Olivier Teulié, *Problèmes diophantiens simultanés*, Monatsh. Math. **143** (2004), no. 3, 229–245, DOI 10.1007/s00605-003-0199-y (French, with English summary). MR **2103807 (2005h:**11147)

[5] Manfred Einsiedler and Dmitry Kleinbock, *Measure rigidity and p-adic Littlewood-type problems*, Compos. Math. **143** (2007), no. 3, 689–702. MR **2330443 (2008f:**11076)

[6] Manfred Einsiedler, Anatole Katok, and Elon Lindenstrauss, *Invariant measures and the set of exceptions to Littlewood's conjecture*, Ann. of Math. (2) **164** (2006), no. 2, 513–560, DOI 10.4007/annals.2006.164.513. MR **2247967 (2007j:**22032)

[7] Wolfgang M. Schmidt, *On badly approximable numbers and certain games*, Trans. Amer. Math. Soc. **123** (1966), 178–199. MR 0195595 (33 #3793)

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